

On Transport and Plasma Flow in Axisymmetric
High-Beta Configurations

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Abstract

A system of differential equations for the flow velocity components V_r , V_θ and V_z in axisymmetric high-beta plasmas is derived from the equilibrium equation and from transport equations. It is proved that the equation for rV_r , used in the Garching high-beta transport code, represents a special case of this system with B_r and the z-derivatives equal to zero. An important result is that the components V_θ and V_z do not enter into the cylindrical slab case.

The local plasma flow is driven by diffusion and by the gradients of heating and loss power. Special cases of the one-dimensional equation are studied analytically and compared with the numerical simulation of the full transport problem.

1. Introduction

It is generally supposed that an economic reactor needs a higher beta than has been achieved in present tokamaks. In future machines larger poloidal beta values will be aimed at by applying considerably increased heating power. In recent years tokamak-like configurations with high beta values have been generated by shock heating. Especially in belt-pinches with strongly elongated cross-sections it has been possible to achieve poloidal beta values equal to the aspect ratio and high-beta periods of about 100 μ s. Consequently, in belt-pinches diffusion processes are growing more and more important, in contrast to other pinch plasmas, which are governed by the dynamics on the fast MHD time scale.

When describing transport processes on the diffusion time scale by fluid models, the flow velocity of the plasma has to be determined. In doing this, the introduction of the fast MHD time scale, e.g. by using inertial forces, must be avoided. Otherwise the calculation will not be able to simulate transport processes for a sufficiently long period of physical time. In the high-beta case the calculation of plasma flow becomes more difficult than it is at low beta because a system of differential equations for the velocity components has to be solved, which is derived from the equilibrium equation and from transport equations. This way had to be followed when the Garching high-beta transport code was developed /1, 2/, which in the meantime has been successfully used in transport investigations of high-beta plasmas in the presence of light impurities (oxygen and carbon) and neutral hydrogen /3, 4, 5/.

In Section 2 we derive the system of differential equations for the flow velocity components in axisymmetric high-beta plasmas. These equations are represented in cylindrical coordinates, so that they can be specialized to the cylindrical slab case modelled by the high-beta transport code. By this procedure the differential equation for the radial velocity component v_r in Ref. /2/ has now been derived from more general grounds and the role of the other velocity components v_θ and v_z made clear. The differential equations for the complete transport problem are rather complicated,

already in one dimension, and need a numerical treatment. In Section 3 we discuss some special cases of the equation for V_r and deal with the role of the loss or heating power on plasma flow, on compressional heating or expansional cooling and on convective heat transfer. Results are compared with numerical solutions of the full one-dimensional transport problem.

2. Flow Velocity in Axisymmetric High-Beta Plasmas

In this section we derive the system of differential equations determining the flow velocity in two-fluid, axisymmetric high-beta plasmas from the equilibrium equation and from transport equations. They are represented in cylindrical coordinates r , θ and z . All three velocity components V_r , V_θ and V_z must be taken into account.

Differentiation of the equilibrium equation

$$\vec{\nabla} p = \frac{1}{c} \vec{j} \times \vec{B}$$

with respect to time yields

$$\vec{\nabla} \frac{\partial p}{\partial t} = \frac{1}{c} \vec{j} \times \frac{\partial \vec{B}}{\partial t} + \frac{1}{c} \frac{\partial \vec{j}}{\partial t} \times \vec{B}$$

By introducing $\vec{D} = \frac{1}{c} \frac{\partial \vec{B}}{\partial t}$ and using Ampere's law

$$\vec{j} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B}$$

one obtains

$$\vec{\nabla} \frac{\partial p}{\partial t} = \vec{j} \times \vec{D} + \frac{c}{4\pi} (\vec{\nabla} \times \vec{D}) \times \vec{B} \quad (1)$$

where $\frac{\partial p}{\partial t}$ can be derived from the sum of the energy equations for electrons and ions:

$$\begin{aligned} \frac{\partial p}{\partial t} = H = & -\vec{\nabla} \cdot (p_e \vec{V}_e) - \vec{\nabla} \cdot (p_i \vec{V}_i) - \frac{2}{3} p_e \vec{\nabla} \cdot \vec{V}_e \\ & - \frac{2}{3} p_i \vec{\nabla} \cdot \vec{V}_i - \frac{2}{3} \vec{\nabla} \cdot (\vec{q}_e + \vec{q}_i) + \frac{2}{3} \eta j^2 + \frac{2}{3} W - \frac{2}{3} P_{rad} \end{aligned}$$

where q_e and q_i describe the heat conduction due to electrons and ions, W is an additional heating power and P_{rad} is composed of bremsstrahlung, impurity line radiation and losses due to ionization of impurities. If problems with constant temperature and thus density on flux surfaces are to be treated, the divergence of surface velocity components can be set equal to zero. As the current density \vec{j}_ψ perpendicular to flux surfaces vanishes, $(V_e)_\psi = (V_i)_\psi$ results, yielding

$$H = -\frac{5}{3}\rho \vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{\nabla} p - \frac{2}{3}\vec{\nabla} \cdot (\vec{q}_e + \vec{q}_i) + \frac{2}{3}\eta j^2 + \frac{2}{3}W - \frac{2}{3}P_{rad} \quad (2)$$

The quantity \vec{D} is given by Faraday's law $\frac{\partial \vec{B}}{\partial t} = -c \vec{\nabla} \times \vec{E}$ and reads

$$\vec{D} = -\vec{\nabla} \times \vec{E}$$

It should be noted that the plasma flow is determined by $\frac{\partial p}{\partial t}$, i.e. by the energy equations, and by magnetic field contributions. The continuity equation only enters implicitly, when the density and temperature variations have to be calculated, too.

In cylindrical coordinates one finds

$$\vec{j} = \frac{c}{4\pi} \left\{ -\frac{\partial B_\theta}{\partial z} \vec{i} - \frac{\partial B_r}{\partial z} \vec{j} - \frac{\partial B_z}{\partial r} \vec{i} + \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) \vec{j} \right\} \quad \text{and}$$

$$\vec{D} = \left\{ \frac{\partial E_\theta}{\partial z} \vec{i} - \frac{\partial E_r}{\partial z} \vec{j} + \frac{\partial E_z}{\partial r} \vec{i} - \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \vec{j} \right\}$$

By substituting $\frac{\partial p}{\partial t}$, \vec{j} and \vec{D} in Eq. (1) the following generally valid system of differential equations is derived:

r-component

$$\frac{4\pi}{c} \frac{\partial H}{\partial r} = - \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) + \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \right) + B_z \left\{ \frac{\partial^2 E_\theta}{\partial z^2} + \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \right] \right\} + B_\theta \frac{1}{r} \frac{\partial}{\partial r} [r \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \right)] \quad (3)$$

θ - component

$$0 = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) \frac{\partial E_\theta}{\partial z} - \frac{\partial B_\theta}{\partial z} \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) - B_r \frac{1}{r} \frac{\partial}{\partial r} [r (\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r})] - B_z \frac{\partial}{\partial z} (\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}) \quad (4)$$

z-component

$$\frac{4\pi}{c} \frac{\partial H}{\partial z} = \frac{\partial B_\theta}{\partial z} (\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}) - (\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}) \frac{\partial E_\theta}{\partial z} + B_\theta \frac{\partial}{\partial z} (\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}) - B_r \left\{ \frac{\partial^2 E_\theta}{\partial z^2} + \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \right] \right\} \quad (5)$$

In order to find the set of equations for V_r , V_θ and V_z the V -dependent terms of H and \vec{E} are needed. From the generalized Ohm's law one obtains in very good approximation $\vec{E} = -\frac{1}{c} \vec{v} \times \vec{B} + \eta \vec{j}$, yielding

$$\vec{E} = \left\{ -\frac{1}{c} (V_\theta B_z - V_z B_\theta) + \eta j_r i - \frac{1}{c} (V_z B_r - V_r B_z) + \eta j_\theta e; -\frac{1}{c} (V_r B_\theta - V_\theta B_r) + \eta j_z e \right\}$$

After substituting H and \vec{E} one arrives at a rather complicated system of second-order differential equations with two space coordinates r and z and three velocity coordinates V_r , V_θ and V_z , which can only be solved numerically.

We now specialize Eqs. (3), (4) and (5) by setting B_r and all z -derivatives equal to zero to the cylindrical slab case, which was modelled by the Garching high-beta transport code. All terms of the θ and z -component vanish and the r -component reads:

$$\begin{aligned} \frac{4\pi}{c} \frac{\partial H}{\partial r} = & \frac{\partial B_z}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) - \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) \frac{\partial E_z}{\partial r} \\ & + B_z \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \right] - B_\theta \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial E_z}{\partial r}) \end{aligned} \quad (6)$$

The expressions for \vec{j} and \vec{E} reduce to

$$\vec{j} = \frac{c}{4\pi} \left\{ 0; -\frac{\partial B_z}{\partial r}; \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) \right\} \quad \text{and}$$

$$\vec{E} = \left\{ -\frac{1}{c} (V_\theta B_z - V_z B_\theta); \frac{1}{c} V_r B_z + \eta j_\theta; -\frac{1}{c} V_r B_\theta + \eta j_z \right\}$$

By inserting H , \vec{j} and \vec{E} into Eq. (6) and by using again the pressure balance relation the following second-order differential equation for $r V_r$ is obtained after a rather lengthy calculation:

$$A \frac{\partial^2}{\partial r^2} (r V_r) - B \frac{\partial}{\partial r} (r V_r) + C r V_r - D = 0 \quad (7)$$

$$A = \frac{5}{3} n (T_e + T_i) + \frac{1}{4\pi} (B_\theta^2 + B_z^2)$$

$$B = \frac{1}{3} \frac{\partial}{\partial r} [n (T_e + T_i)] + A \frac{1}{r} + \frac{1}{2\pi} B_\theta^2 \frac{1}{r}$$

$$C = \frac{1}{2\pi} B_\theta \frac{1}{r} \left(B_\theta \frac{1}{r} - \frac{\partial B_\theta}{\partial r} \right)$$

$$D = r \frac{\partial E}{\partial r} + 2P$$

$$E = -\frac{2}{3} \ell_2 n^2 - \frac{2}{3} \ell_3 n + \frac{2}{3} \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\kappa_1 T_e^2 \frac{\partial T_e}{\partial r} + \kappa_2 T_i^2 \frac{\partial T_i}{\partial r} \right) \right]$$

$$+ \frac{2}{3} \left(\frac{c}{4\pi} \right)^2 \eta \left\{ \left[\frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) \right]^2 + \left(\frac{\partial B_z}{\partial r} \right)^2 \right\} + \frac{2}{3} W$$

$$+ P + \left(\frac{c}{4\pi} \right)^2 B_z \frac{1}{r} \frac{\partial}{\partial r} \left(\eta r \frac{\partial B_z}{\partial r} \right)$$

$$P = \left(\frac{c}{4\pi}\right)^2 B_\theta \frac{\partial}{\partial r} \left[\eta \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) \right]$$

Apart from neutral hydrogen terms of the three-fluid model, which have not been taken into account here, equation (7) is identical to the differential equation for rV_r given in Ref. /2/. Thus, it has been proved that this equation is a one-dimensional special case of the general axisymmetric problem. It represents the only equation to be solved because the θ and z -components (Eqs. (4) and (5)) vanish. Obviously, the surface velocity components V_θ and V_z do not enter in one-dimensional geometry because they only occur in expressions with $B_r (= 0)$ and z -derivatives ($= 0$) as factors. V_θ and V_z are arbitrary and must not be calculated, which means a considerable simplification compared with the two-dimensional case.

It should be mentioned that the general axisymmetric problem can be treated in flux-surface coordinates ψ , θ and χ , as well /6/. A very difficult non-standard equation that contains flux-surface averages is obtained from which, in principle, V_ψ can be determined. In flux-surface coordinates the expression $r h_3 V_\psi$ occurs with h_3 being the metric coefficient of the χ -coordinate, which for cylindrical flux surfaces with $h_3 = 1$ and $V_\psi = V_r$ reduces to the rV_r in Eq. (7).

3. Discussion of One-Dimensional Plasma Flow

The complete transport problem of high-beta plasmas has been solved numerically in one dimension. In order to get relations that can be used to judge more complicated, e.g. two-dimensional, situations, we shall discuss several special cases of Eq. (7) for rV_r and simulation experiments for the standard case of Belt-Pinch IIa.

If one assumes $D = 0$ in Eq. (7) the trivial solution $rV_r = 0$ results, which means that the plasma is at rest in the region with $D = 0$. Consequently, the only driving

forces for plasma flow occur in $D = r \frac{\partial E}{\partial r} + 2 P$, namely diffusion and the gradient of heating and loss power. When there are no gradients of radiation losses, heat conduction losses and additional heating power present, the plasma flow is caused by diffusion and ohmic heating only. On the other hand, in the ideal MHD case ($\eta = 0$) the flow of plasma results from gradients of heating and loss power alone, the larger velocities corresponding to the steeper gradients. This is the typical situation of present belt-pinch plasmas, where the magnetic field diffusion is relatively slow and the impurity radiation losses are dominant /4, 5/. It is obvious from Eqs. (1) and (2) that also two-dimensional velocity fields can be interpreted in terms of the gradients of heating and loss power. The cylindrical slab model with a pressure profile and a radial gradient of radiation losses equal to those in the midplane of Belt-Pinch IIa should yield roughly the radial velocity distribution of the experiment. In two dimensions, however, the axial gradient of the loss power must drive an additional V_z flow, which changes its direction in the midplane and which results in a stronger compression than in the one-dimensional case. Indeed, this is found when comparing the density compression near the centre predicted by the high-beta transport model with the measured density increase in Belt-Pinch IIa /5/.

Simulation experiments for belt-pinches and hotter high-beta plasmas have shown that the velocity field of plasma flow is indeed determined by the space dependence of the loss power due to impurities P_{rad} . At the beginning P_{rad} exhibits a maximum near the magnetic axis and the plasma flow is directed to this region (see Fig. 1). This situation is maintained until the radiation losses drop sharply owing to the ionization of O VI in the centre to the helium-like state. Then, a hollow P_{rad} profile develops, the gradient of P_{rad} becomes smaller than it is for the other loss processes and the plasma flow is no longer directed to the central region.

The next special case to be discussed has a beta equal to one, corresponding to vanishing magnetic fields B_θ and B_z and to a constant pressure p_0 . If heat

conduction, but not convective heat transfer, is neglected in Eq. (7) and if there is no additional heating source, one obtains

$$A = \frac{5}{3} p_0, B = \frac{5}{3} p_0 \frac{1}{r}, C = 0, P = 0, D = r \frac{\partial E}{\partial r} \quad \text{and}$$

$$E = -\frac{2}{3} (\mathcal{L}_2 n^2 + \mathcal{L}_3 n) = -\frac{2}{3} P_{\text{rad}} \quad \text{yielding}$$

$$p_0 \frac{\partial^2}{\partial r^2} (r v_r) - p_0 \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{2}{5} r \frac{\partial}{\partial r} P_{\text{rad}} = 0 \quad (8)$$

This equation can also be derived directly from Eq. (6), i.e. from $\frac{\partial H}{\partial r} = 0$. It can be solved for a special case $P_{\text{rad}} = c_1/r$, which has been found in a region around the magnetic axis by the transport code calculations. c_1 is a positive constant. The solution reads

$$r v_r = -\frac{2}{5} \frac{c_1}{p_0} r + c_2$$

It is interesting that in the simulation of the full problem this linear r - dependence of $r v_r$ is found (see Fig. 1) and that it occurs in the range where P_{rad} is given by c_1/r . Increasing the radiation power and thus c_1 results in a steeper slope. At first sight it is surprising that the highly simplified $\beta = 1$ model yields these good results. An explanation is given by an expansion of Eq. (7) around the axis with redistric fields but without diffusion and heat conduction, which shows that the $C r v_r$ - term may be neglected. Thus, the $\beta = 1$ case with $C = 0$ is not a bad approximation.

The second term in Eq. (8) represents the compressional heating or expansional cooling $P_{\text{com}} = -p \vec{\nabla} \vec{v}$. Another velocity dependent expression in the energy equation is convective heat transfer $P_{\text{conv}} = -\vec{\nabla} \left(\frac{3}{2} p \vec{v} \right)$, which for $\beta = 1$ is equal to $3/2 P_{\text{com}}$. With $P_{\text{rad}} = c_1/r$ in Eq. (8) it is found that P_{com} amounts to 40 % and P_{conv} to 60 % of the radiation power. These values are

upper limits that cannot be reached in the general case, where the magnetic field has to be compressed together with the plasma. The numerical treatment of the complete problem for Belt-Pinch IIa resulted in a P_{com} of typically 20 % of P_{rad} (see Fig. 2). Generally, it holds that $P_{\text{conv}} = \frac{3}{2} P_{\text{com}} - \frac{3}{2} \vec{v} \vec{\nabla} \rho$ the second term being negative or zero. P_{conv} reaches $3/2 P_{\text{com}}$ only at the magnetic axis ($\vec{\nabla} \rho = 0$) or at a position with vanishing V_{ψ} .

The relation $P_{\text{com}} / P_{\text{rad}}$ from a simulation experiment for the standard case of Belt-Pinch IIa is plotted as a function of r for $t = 40$ and $64 \mu\text{s}$ in Fig. 2. The dashed curve is obtained for $t = 40 \mu\text{s}$ by the above described expansion near the axis for general beta and without heat conduction. One can conclude that the weak r -dependence of $P_{\text{com}} / P_{\text{rad}}$ in the full problem results from heat conduction that has been shown to be important in the local energy balance /5/. It is very likely that the approximate proportionality in the central plasma region between the compressional heating power and the power loss due to impurities is also present in two dimensions. A corresponding expansional cooling is produced by a source of additional heating power.

4. Summary

The plasma flow in axisymmetric high-beta configurations has been determined from the pressure balance and from transport equations. A system of three differential equations for the flow velocity components V_r , V_{θ} and V_z has been derived. It has been proved that the differential equation for rV_r used in the Garching high-beta transport code /2/ represents a special case ($B_r = 0$ and vanishing z -derivatives) of the general system of equations. The velocity components V_{θ} and V_z do not enter the one-dimensional problem and must not be calculated. A general result, valid also in two dimensions, is that the plasma flow is driven by the gradients of heating and loss power and by diffusion, the larger velocities corresponding to steeper gradients and higher resistivity. For a given profile of loss power a somewhat stronger compression is expected in two dimensions.

The special case with $\beta = 1$, no heat conduction and no additional heating source has been analyzed. For a radiative power of the form $P_{\text{rad}} = C_1/r$ a solution for rV_r is obtained that linearly depends on r in agreement with the simulation results. Under these conditions compressional heating power P_{com} amounts to 40 % and convective energy transfer P_{conv} to 60 % of P_{rad} . Both values for the field-free case are upper limits because in the general case the magnetic field has to be compressed together with the plasma. Simulation experiments have shown that $P_{\text{com}} / P_{\text{rad}}$ has a weak r -dependence, which results from heat conduction. One can conclude that P_{com} should be roughly proportional to P_{rad} also in two-dimensional geometry.

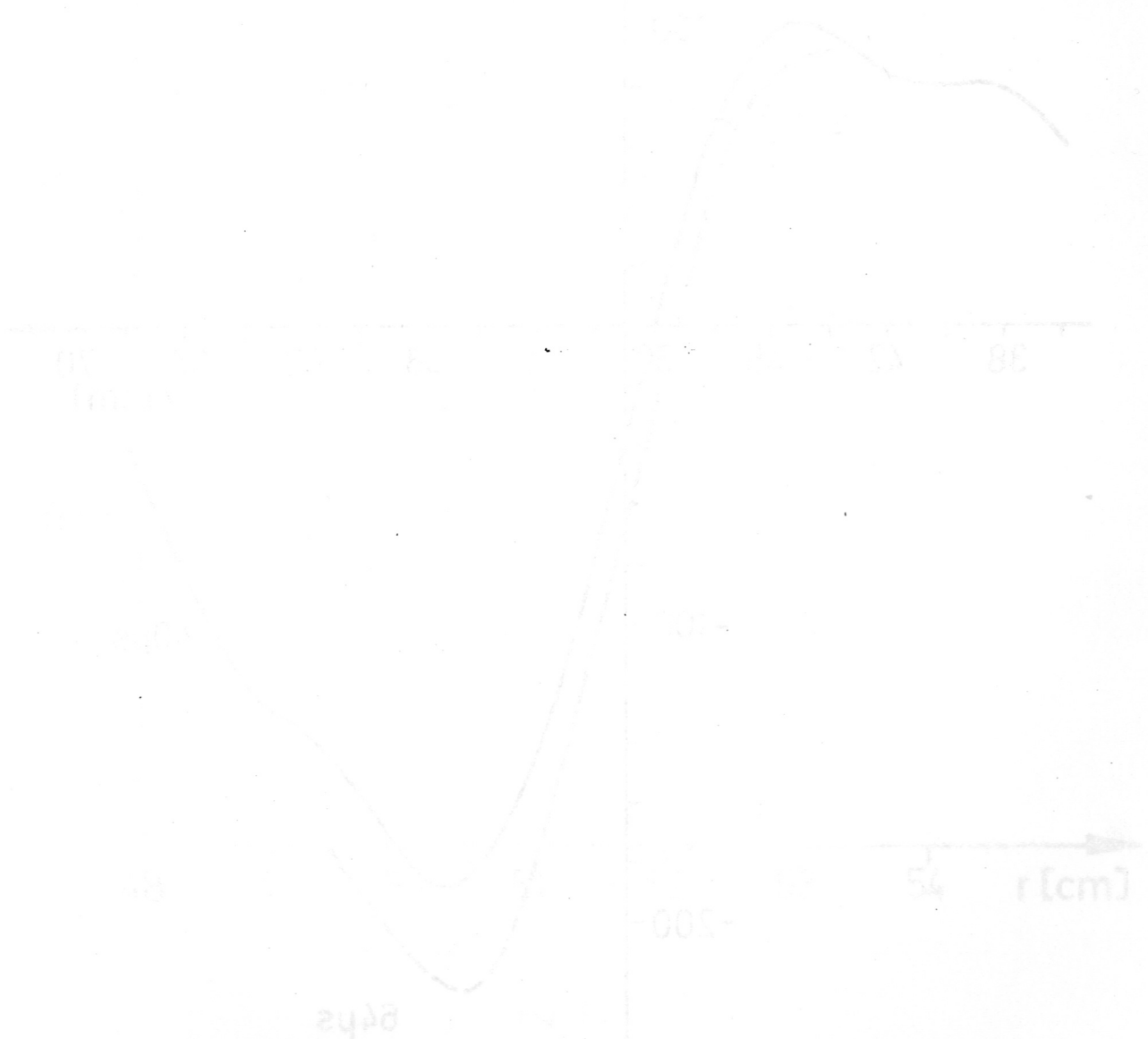
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Figure Captions

Fig. 1 Computed rV_r - profiles in the standard case of Belt-Pinch IIa.

Fig. 2 Weak r -dependence of the ratio of compressional heating power P_{com} to impurity loss power P_{rad} in the central plasma region.



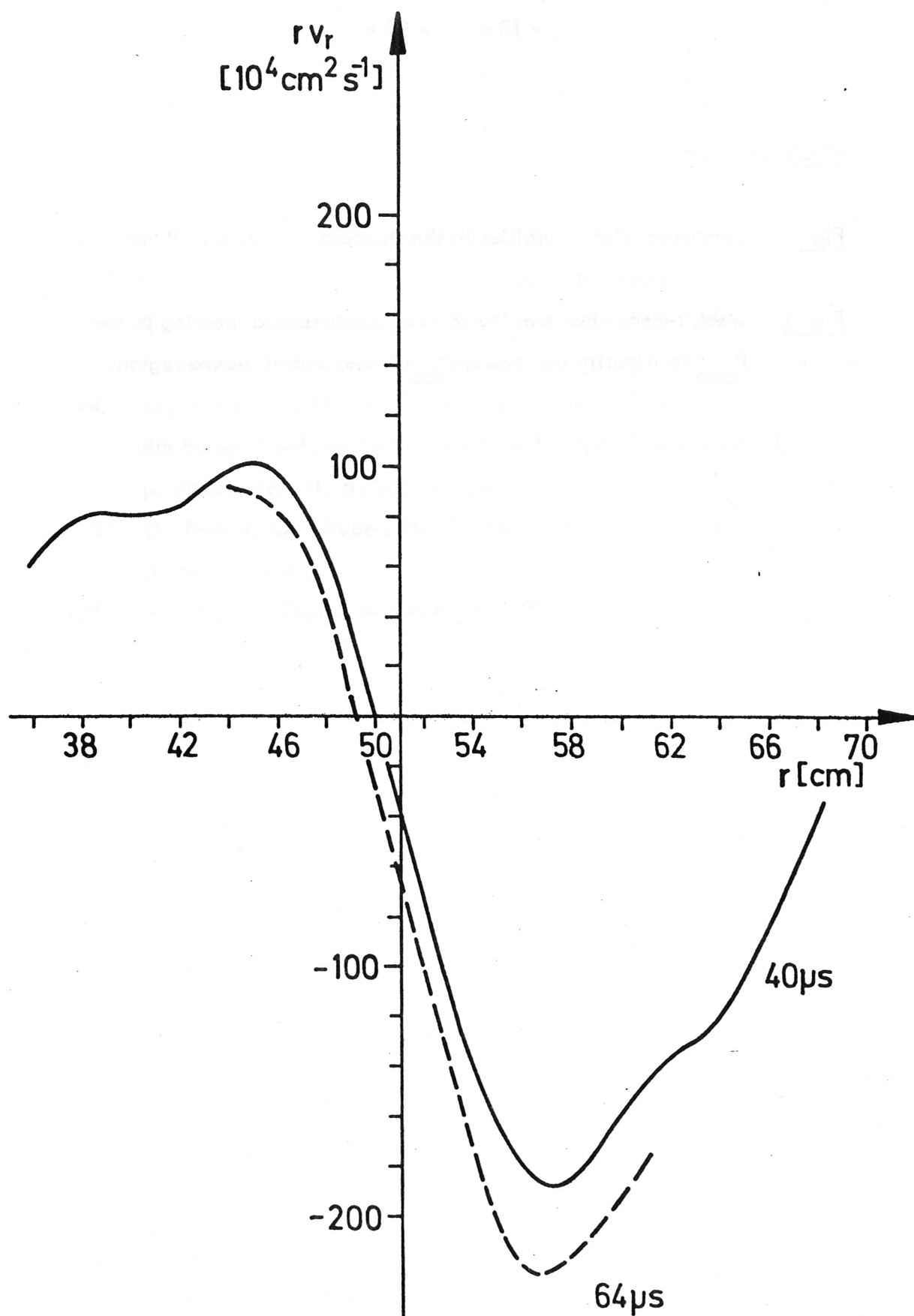


Fig. 1

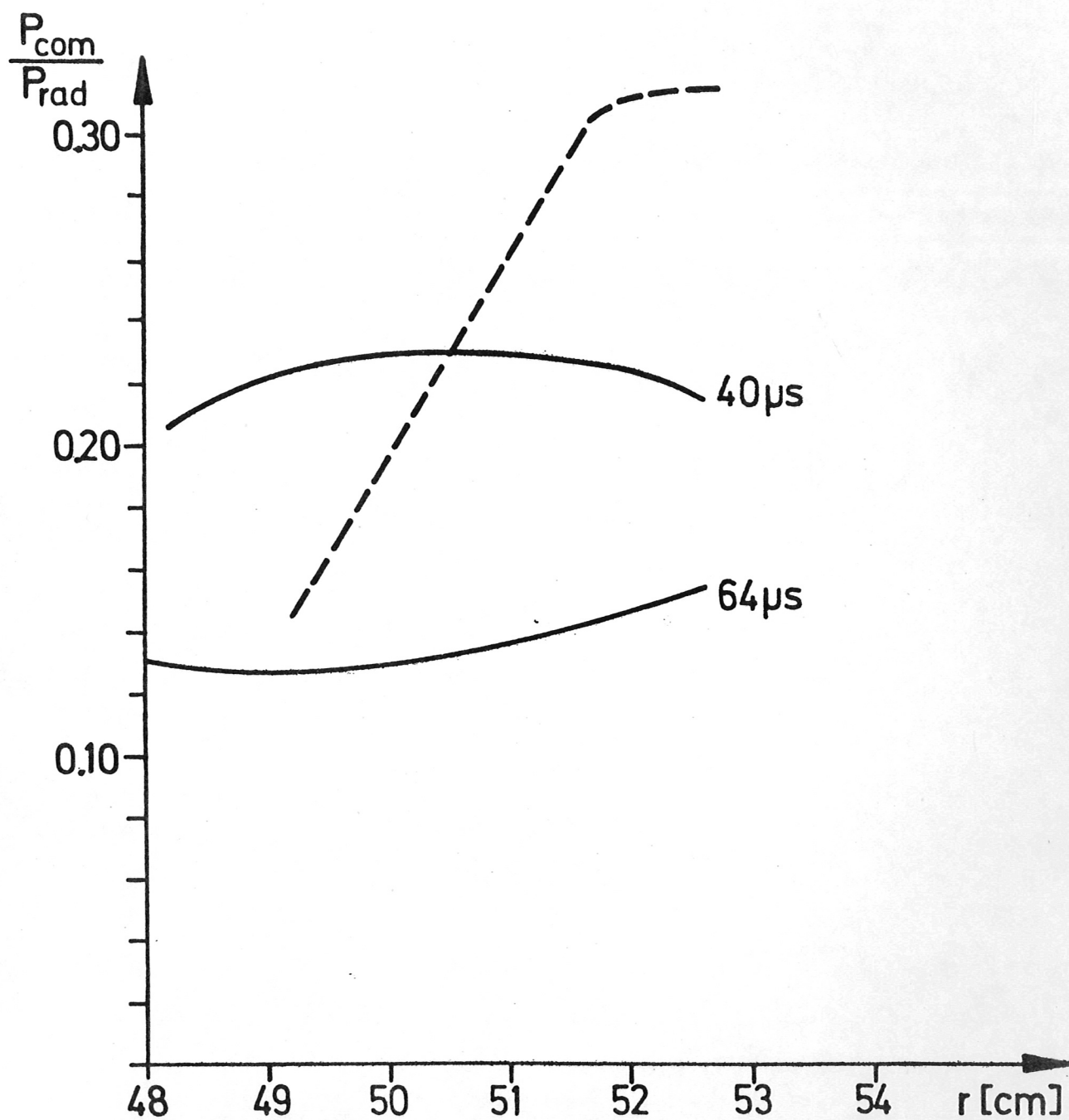


Fig. 2